Surfaces in finite covers and the group determinant

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(joint work with Daryl Cooper)

Let Y be a 3-manifold with one torus boundary component and $\tilde{Y} \to Y$ a finite regular covering of Y induced by a map $\pi_1(Y) \to G$. Let $K(\tilde{Y})$ denote the kernel of the map $i_* : H_1(\partial\tilde{Y}; \mathbb{Q}) \to H_1(\tilde{Y}; \mathbb{Q})$. For a slope α on ∂Y, denote by $V(\alpha, \tilde{Y})$ the subspace of $H_1(\partial\tilde{Y}; \mathbb{Q})$ spanned by pre-images of α. We say that a slope α on ∂Y is a *virtual homology slope* (for $\tilde{Y}$ of rank n) if $dim(K(\tilde{Y}) \cap V(\alpha, \tilde{Y})) = n > 0$. Note that if α is a virtual homology slope, then there is a non-separating surface in $\tilde{Y}$ whose boundary components map down to curves that have slope (a multiple of) α in ∂Y.

We relate the group determinant of G, as studied by Frobenius and Dedekind, to a matrix that encodes $K(\tilde{Y})$. This enables us to prove the following theorems:

Theorem 0.1. *Let Y be a hyperbolic 3-manifold with one torus boundary component. Then either:*
i) for all $n \in \mathbb{N}$, there is a regular cover $\tilde{Y} \to Y$ and a slope α on ∂Y so that α is a virtual homology slope for $\tilde{Y}$ of rank at least n, or
ii) every slope on ∂Y is a virtual homology slope.

Theorem 0.2. *Let Y be a hyperbolic 3-manifold with one torus boundary component. Then infinitely many fillings of Y are virtually Haken.*

Theorem 0.2 is implied for the case that Y is not fibered by [2].

Let G be a finite group, and $R : G \to Aut(\mathbb{C}^{|G|})$ the representation induced by the right regular action of G on itself, $g : h \mapsto hg^{-1}$. Let $\{X_g\}$ be a collection of commuting variables, one for each element of G. For any representation ρ of G, the representation matrix $M(\rho)$ is the matrix $\sum_{g\in G} \rho(g)X_g$. Then the group matrix of G, $M(G)$, is the representation matrix for R. Thus $M(G)$ has $X_{g_i^{-1}g_j}$ as the ij-th entry. The *group determinant* of G is $det(M(G))$. Important to our computations is the fact that if a representation is reducible, i.e., $\rho = \rho_1 \oplus \rho_2$, then the representation determinant $det(\rho) = det(M(\rho))$ is a product $det(\rho_1)det(\rho_2)$. Since we will ultimately be interested in linear factors of the group determinant, we will look for irreducible representations that have linear factors in their determinants.

In our applications we will have a matrix that is a specialization of the group matrix that is symmetric. Thus to simplify matters we work with the symmetrized group matrix, $M^{sym}(G)$ which is obtained from $M(G)$ by setting $X_g = X_{g^{-1}}$. Similarly $det^{sym}(G) = det(M^{sym}(G))$. For example, the group determinant of $\mathbb{Z}_3$ is $(a + b + c)(a^2 - ab + b^2 - ac - bc + b^2)$, while $det^{sym}(\mathbb{Z}_3) = (a + 2b)(a - b)^2$.

Given a regular cover $\tilde{Y} \to Y$ induced by a map $\pi_1(Y) \to G$, we define a matrix $B(\tilde{Y})$ that encodes the vector space $K(\tilde{Y})$. Rational eigenvalues of $B(\tilde{Y})$ are virtual homology slopes, and the dimension of the associated eigenspace equals the rank of the slope. When $\pi_1(\partial Y) \to 1 \in G$, the boundary matrix is a specialization of $M^{sym}(G)$. When $det(M^{sym}(G))$ has a rational linear factor of rank n, any regular covering $\tilde{Y} \to Y$ with covering group G where the boundary torus lifts will have a virtual homology slope of rank at least n.

In general however, the boundary torus will not lift, and in this case we identify the variables X_{g_1} and X_{g_2} whenever g_1 and g_2 are in the same element of $\{HgH \cup Hg^{-1}H\}_{g \in G}$. This yields the group matrix of G with respect to H, $M(G,H)$, and we can determine the rational eigenvalues of $B(\tilde{Y})$ from this matrix.

The covers we use are induced by the surjections to $PSL(2,\mathbb{F}_p)$ given in [5]. This implies that $\pi_1(Y)$ surjects $PSL(2,\mathbb{F}_p)$ for infinitely many p where $\pi_1(\partial M)$ maps onto the cyclic subgroup of order p. By analyzing the permutation representation induced by the action of $PSL(2,\mathbb{F}_p)$ on $\mathbb{P}^1(\mathbb{F}_p)$, we show that there is an invariant slope of rank at least p for any such cover. This is the idea of the proof of Theorem 0.1. An application of [4] shows that if $p \geq 2$, there is a non-separating surface S in $\tilde{Y}$ that is not the fiber of a fibration and whose boundary curves all map down to curves of the same slope in ∂Y. By results in [6], [2] and [1], a large enough cyclic cover of $\tilde{Y}$ dual to S will contain a closed incompressible surface, a lift of Freedman-Freedman tubing of two copies of S, and this will survive fillings that are distance greater than 1 from this slope. Since infinitely many fillings of Y lift to this cover, this implies Theorem 0.2.

References

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