Virtually Haken fillings and semi-bundles

Daryl Cooper and Genevieve S. Walsh

Abstract

Suppose that M is a fibered three-manifold whose fiber is a surface of positive genus with one boundary component. Assume that M is not a semi-bundle. We show that infinitely many fillings of M along ∂M are virtually Haken. It follows that infinitely many Dehn-surgeries of any non-trivial knot in the three-sphere are virtually Haken.

1 Introduction

In this paper *manifold* will always mean a compact, connected, orientable, possibly bounded, three-manifold. A *bundle* means a manifold which fibers over the circle. A *semi-bundle* is a manifold which is the union of two twisted I-bundles (over connected surfaces) whose intersection is the corresponding ∂I-bundle. An irreducible, ∂-irreducible manifold that contains a properly embedded incompressible surface is called *Haken*. A manifold is *virtually Haken* if has a finite cover that is Haken.

Waldhausen's *virtually Haken conjecture* is that every irreducible closed manifold with infinite fundamental group is virtually Haken. It was shown in [**?**] that *most* Dehn-fillings of an atoroidal Haken manifold with torus boundary are virtually Haken provided the manifold is not a bundle.

Theorem 1. *Suppose that M is a bundle with fiber a compact surface F and that F has exactly one boundary component. Also suppose that M is not a semi-bundle and not $S^1 \times D^2$. Then infinitely many Dehn-fillings of M along ∂M are virtually Haken.*

Corollary 2. *Let k be a knot in a homology three-sphere N. Suppose that $N - k$ is irreducible and that k does not bound a disk in N. Then infinitely many Dehn-surgeries along k are virtually Haken.*

The main idea is to construct a surface of *invariant slope* (see section 3) in a particular finite cover of M. Such surfaces are studied in arbitrary covers using representation theory in a sequel [**?**]. While writing this paper we noticed that Thurston's theory of bundles extends to semi-bundles, and in particular there are manifolds which are semi-bundles in infinitely many ways. We discuss this in the next section.

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2 Bundles and Semi-Bundles

Various authors have studied semi-bundles, in particular [**?**], [**?**],[**?**]. Suppose a manifold has a regular cover which is a surface bundle. We wish to know when a particular fibration in the cover corresponds to a bundle or semi-bundle structure on the quotient. The following has the same flavor as some results of Hass in [**?**].

Theorem 3. *Let M be a compact, connected, orientable, irreducible three-manifold, $p:\tilde{M}\to M$ a finite regular cover, and G the group of covering automorphisms. Suppose that $\phi:\tilde{M}\to S^1$ is a fibration of $\tilde{M}$ over the circle. Suppose that the cyclic subgroup V of $H^1(\tilde{M};\mathbb{Z})$ generated by $[\phi]$ is invariant under the action of G. Then one of the following occurs:*

1. *The action of G on V is trivial. Then M also fibers over the circle. Moreover there is a fibering of M which is covered by a fibering of $\tilde{M}$ that is isotopic to the original fibering.*
2. *The action of G on V is non-trivial. Then M is a semi-bundle. Moreover there is a semi-fibering of M which is covered by a fibering of $\tilde{M}$ that is isotopic to the original fibering.*

Proof. Define $N=\ker[\phi_*:\pi_1\tilde{M}\to\pi_1 S^1]$. Since ϕ is a fibration N is finitely generated. If N is cyclic then the fiber is a disc or annulus. In these cases the result is easy. Thus we may assume N is not cyclic. Because V is G-invariant, it follows that N is a normal subgroup of $\pi_1 M$ and $Q=\pi_1 M/N$ is infinite. Using [**?**, Theorem 3] it follows that M is a bundle or semi-bundle (depending on case 1 or 2) with fiber a compact surface F and N has finite index in $\pi_1 F$. The pull-back of this (semi)fibration of M gives a fibration of $\tilde{M}$ in the cohomology class of ϕ and is therefore isotopic to the given fibration. $\square$

Suppose that $G\cong(\mathbb{Z}_2)^n$ acts on a real vector space V and let $X=Hom(G,\mathbb{C})$ denote the set of characters on G. Then $X\cong Hom(G,\mathbb{Z}_2)$. For each $\epsilon\in X$ there is a G-invariant *generalized ϵ-eigenspace*

$$V_\epsilon=\{\ v\in V\ :\ \forall g\in G\quad g\cdot v=\epsilon(g)v\ \}.$$

Then V is the direct sum of these subspaces V_ϵ.

Suppose that M is an atoroidal irreducible manifold with boundary consisting of incompressible tori. According to Thurston there is a finite collection (possibly empty), $\mathcal{C}=\{C_1,\cdots,C_k\}$, called *fibered faces*. Each fibered face is the interior of a certain top-dimensional face of the unit ball of the Thurston norm on $H_2(M,\partial M;\mathbb{R})$. It is an open convex set with the property that fibrations of M correspond to rational points in the projectivized space $\mathbb{P}(\cup_i C_i)\subset\mathbb{P}\left(H_2(M,\partial M;\mathbb{R})\right)$.

Let $G=H_1(M;\mathbb{Z}/2)$. The regular cover $\tilde{M}_s$ of M with covering group G is called the *$\mathbb{Z}_2$-universal cover.* Let $\mathcal{D}=\{D_1,\cdots,D_l\}$ be the fibered faces for this cover. For each $\epsilon\in H^1(M;\mathbb{Z}_2)$ there is an ϵ-eigenspace $H_{2,\epsilon}$ of $H_2(\tilde{M}_s,\partial\tilde{M}_s;\mathbb{R})$. For each $1\le i\le l$ and $\epsilon\in H^1(M;\mathbb{Z}_2)$ we call $S_{i,\epsilon}=D_i\cap H_{2,\epsilon}$ a *semi-fibered face* if it is not empty. It is the interior of a compact convex polyhedron whose interior is in the interior of some fibered face for $\tilde{M}_s$. Let S_i be the union of the $S_{i,\epsilon}$ where ϵ is non-trivial.

Theorem 4. *With the above notation there is a bijection between isotopy classes of semi-fiberings of M and rational points in $\mathbb{P}(\cup_i S_i)$.*

Proof. A semi-fibration of M gives such a rational point by considering the induced fibration on $\tilde{M}_s$. The converse follows from Theorem 3. We leave it as an exercise to check uniqueness up to isotopy. □

We believe that all points in $\mathbb{P}(\cup_i S_i)$ correspond to isotopy classes of non-transversally-orientable, transversally-measured, product-covered 2-dimensional foliations of M. This is true for rational points and therefore holds on a dense open set (using the fact that the set of non-degenerate twisted 1-forms is open). However, since we have no use for this fact, we have not tried very hard to prove it.

Definition. A manifold is a *sesqui-bundle* if it is both a bundle and a semi-bundle.

An example is the torus bundle M with monodromy -Id. This is the quotient of Euclidean three-space by the group $\mathcal{G}_2$ in [**?**, Theorem 3.5.5]. M has infinitely many semi-fibrations with generic fiber a torus and two Klein-bottle fibers. In addition, M is a bundle thus a sesqui-bundle.

A hyperbolic example may be obtained from M as follows. Let C be a 1-submanifold in M which is a small C^1-perturbation of a finite set of disjoint, immersed, closed geodesics in M chosen so that:

(1) no two components of C cobound an annulus and no component bounds a Mobius strip

(2) C intersects every flat torus and flat Klein bottle.

(3) Each component of C is transverse to both a chosen fibration and semi-fibration.

Let N be M with a regular neighborhood of C removed. Then the interior of N admits a complete hyperbolic metric. By (3) it is a sesqui-bundle. This answers a question of Zulli who asked in [**?**] if there are non-Seifert 3-manifolds which are sesqui-bundles.

3 Virtually Haken Fillings

The following is well-known, but we include it here for ease of reference.

Lemma 5. *Suppose M is Seifert fibered and has one boundary component . Then one of the following holds:*

(1) M is $D^2 \times S^1$ or a twisted I-bundle over the Klein bottle.

(2) Infinitely many Dehn-fillings are virtually Haken.

Proof. The base orbifold Q has one boundary component and no corners. If $\chi^{orb}Q > 0$ then Q is a disc with at most one cone point thus $M = D^2 \times S^1$. If $\chi^{orb}Q = 0$ then Q is a Mobius band or a disc with two cone points labeled 2 and in either case Q has a 2-fold orbifold-cover that is an annulus A. But then M is 2-fold covered by a circle bundle over A. Since M is orientable it follows that this bundle is $S^1 \times A$ and hence M is a twisted I-bundle over the Klein bottle.

Finally, if $\chi^{orb}(Q) < 0$ then all but one filling of M is Seifert fibered. There are infinitely many fillings of M which give a Seifert fibered space, P, with base orbifold Q' and $\chi^{orb}(Q') < 0$. There is an orbifold-covering of Q' which is a closed surface of negative Euler characteristic. The induced covering of P contains an essential vertical torus and is therefore virtually Haken. □

Definitions. A *slope* on a torus T is the isotopy class of an essential simple closed curve on T. We say that a slope *lifts* to a covering of T if it is represented by a loop which lifts. The following is immediate:

Lemma 6. *Suppose $\tilde{T} \to T$ is a finite covering. Then the following are equivalent:*
(1) Some slope on T lifts to $\tilde{T}$.
(2) The covering is finite cyclic.
(3) Infinitely many slopes on T lift to $\tilde{T}$.

The *distance,* $\Delta(\alpha, \beta)$, between slopes α, β on T is the minimum number of intersection points between representative loops. If α is a slope on a torus boundary component of M then $M(\alpha)$ denotes the manifold obtained by Dehn-filling M using α. A surface S in a manifold M is *essential* if it is compact, connected, orientable, incompressible, properly-embedded, and not boundary-parallel. Let M be a manifold with boundary a torus and $\alpha \subset \partial M$ a slope. Suppose that N is a finite cover of M. An essential surface $S \subset N$ has *invariant slope* α if $\partial S \neq \phi$ and every component of ∂S projects to a loop homotopic to a non-zero multiple of α. We call a finite cover $p : N \to M$ a *∂-cover* if there is an integer $d > 0$ and a homomorphism $\theta : \pi_1(\partial M) \to \mathbb{Z}_d$ such that for every boundary component T of N we have $p_*(\pi_1 T) = \ker \theta$. The existence of θ ensures each component of ∂N is the same cyclic cover of ∂M.

The following lemma reduces the proof of the main theorem to constructing an essential non-fiber surface of invariant slope in a ∂-cover of M.

Lemma 7. *Suppose that M is a compact, connected, orientable irreducible 3-manifold with one torus boundary component. Suppose that there is a ∂-cover N of M and an essential non-separating surface $S \subset N$ of invariant slope. Assume that S is not a fiber of a fibration of N. Then M has infinitely many virtually-Haken Dehn-fillings.*

Proof. We first remark that the particular case that concerns us in this paper is that M is a bundle with boundary and thus M is irreducible. Since M is irreducible at most 3 fillings give reducible manifolds, [**?**]. A cover of an irreducible manifold is irreducible [**?**]. Therefore it suffices to show there are infinitely many fillings of M which have a finite cover containing an essential surface.

If M contains an essential torus then this torus remains incompressible for infinitely many Dehn-fillings by [**?**, Theorem 2.4.2]. If M is Seifert fibered then by Lemma 5 either the result holds or $M = S^1 \times D^2$ or is a twisted I-bundle over the Klein bottle. The latter two possibilities do not contain a surface S as in the hypotheses. By Thurston's hyperbolization theorem we are reduced to case that M is hyperbolic.

Since $p : N \to M$ is a ∂-cover there is $d > 0$ such that every component of ∂N is a d-fold cover of ∂M. Let k be a positive integer coprime to d. Let $p_k : \tilde{N}_k \to N$ be the

k-fold cyclic cover dual to S. We claim that there is a homomorphism $\theta_k : \pi_1 M \to \mathbb{Z}_{kd}$ such that every slope in $\ker \theta_k$ lifts to every component of $\partial \tilde{N}_k$.

Assuming this, the filling $M(\gamma)$ of M is covered by a filling, $\tilde{N}_k(\gamma)$, of $\tilde{N}_k$ if and only if the slope $\gamma \subset \partial M$ lifts to each component of $\partial \tilde{N}_k$. Since S is non-separating, by [**?**, Theorem 5.7] , there is $K > 0$ such that if $k \geq K$ then there is an essential closed surface $F_k \subset \tilde{N}_k$ obtained by Freedman tubing two lifts of S. We choose such k coprime to d. By [**?**, Theorem 5.3], there is a finite set of slopes $\beta_1, \cdots, \beta_n$ on ∂M and $L > 0$ so that if $\gamma \subset \partial M$ is a slope and $\Delta(\gamma, \beta_i) \geq L$ for all i then the projection of F_k into $M(\gamma)$ is π_1-injective. Assuming the claim, there are infinitely many slopes $\gamma \in \ker \theta_k$ satisfying these inequalities. For such γ the cover $\tilde{N}_k(\gamma) \to M(\gamma)$ contains the essential surface F_k.

It only remains to prove the claim. Let T be a component of ∂N and $\beta \subset T$ be the slope given by $S \cap T$. Let $\tilde{T}$ be a component of $\partial \tilde{N}_k$ which covers T. The cover $p_k| : \tilde{T} \to T$ is cyclic of degree k' some divisor of k (depending only on $|S \cap T|$). Also β lifts to this cover. Suppose that a slope $\gamma \subset \partial M$ lifts to a slope $\tilde{\gamma} \subset T$. It follows that $\tilde{\gamma}$ lifts to $\tilde{T}$ if k' divides $\Delta(\tilde{\gamma}, \beta)$. If this condition is satisfied by some lift, $\tilde{\gamma}$, of γ then, since S has invariant slope and $N \to M$ is a ∂-cover, it is satisfied by every such lift.

Let $\tilde{T} \to T$ be the k'-fold cyclic cover dual to β. Since k' and d are coprime the composite of this cover and the cyclic d-fold cover $T \to \partial M$ is a cyclic cover of degree dk'. By Lemma 6 there are infinitely many slopes on ∂M which lift to $\tilde{T}$. Every slope on ∂M which lifts to $\tilde{T}$ also lifts to every component of $\partial \tilde{N}_k$. This proves the claim. $\square$

Proof of Theorem 1. We attempt to construct S and N as in Lemma 7. The action of the monodromy on $H_1(F; \mathbb{Z}_2)$ has some finite order m. Therefore there is a finite cyclic m-fold cover $W \to M$ such that W is a bundle with fiber F and the action of the monodromy for W on $H_1(F; \mathbb{Z}_2)$ is trivial. We then have

$$H^1(W; \mathbb{Z}_2) \cong H^1(F; \mathbb{Z}_2) \oplus H^1(S^1; \mathbb{Z}_2).$$

Since F has boundary and $F \neq D^2$ we may choose a non-zero element $\phi = (b, 0) \in H^1(F; \mathbb{Z}_2) \oplus H^1(S^1; \mathbb{Z}_2)$. This determines a two-fold cover $\tilde{W}$ of W. Since F has one boundary component, ϕ vanishes on $H_1(\partial W; \mathbb{Z}_2)$, and since W has one boundary component, $\tilde{W}$ has exactly two boundary components T_1 and T_2. The action of the covering involution, τ, swaps these tori. In particular $\tilde{W} \to M$ is a ∂-cover.

We claim that there is an essential surface S in $\tilde{W}$ such that

$$\tau_*[S] = -[S] \neq 0 \in H_2(\tilde{W}, \partial \tilde{W}; \mathbb{Z}).$$

Using real coefficients, all cohomology groups have direct-sum decomposition into ± 1 eigenspaces for τ^*; thus $H^1(\partial \tilde{W}; \mathbb{R}) = V_+ \oplus V_-$. Since τ swaps T_1 and T_2 then, with obvious notation, it swaps μ_1 with μ_2 and λ_1 with λ_2. If $\epsilon = \pm 1$ then V_ϵ has basis $\{\mu_1 + \epsilon\mu_2, \lambda_1 + \epsilon\lambda_2\}$ and thus has dimension 2. Let

$$K = Im\left[incl^* : H^1(\tilde{W}; \mathbb{R}) \to H^1(\partial \tilde{W}; \mathbb{R})\right].$$

Decompose $K = K_+ \oplus K_-$. We claim that $dim(K_+) = dim(K_-) = 1$. Since $dim(K) = 2$ the only other possibilities are that $K_+ = V_+$ or $K_- = V_-$. The intersection pairing on $\partial\tilde{W}$ is dual to the pairing on $H^1(\partial\tilde{W},\mathbb{R})$ given by $<\phi,\psi> = (\phi\cup\psi)\cap[\partial\tilde{W}]$. This pairing vanishes on K. Since $<\mu_1+\epsilon\mu_2,\lambda_1+\epsilon\lambda_2> = 2<\mu_1,\lambda_1> = \pm 2$, the restriction of $<,>$ to each of $V_\pm$ is non-degenerate. This contradicts $K = V_\pm$.

Choose a primitive class $\phi\in H^1(\tilde{W};\mathbb{Z})$ with $incl^*\phi\in K_-$. Let S be an essential oriented surface in $\tilde{W}$ representing the class Poincaré dual to ϕ. Then $\tau_*[S] = -[S]$ as required.

The 1-manifold $\alpha_i = T_i\cap\partial S$ with the induced orientation is a 1-cycle in $\partial\tilde{W}$. Then $[\partial S] = [\alpha_1]+[\alpha_2]\in H_1(\partial\tilde{W})$. Since T_i is a torus all the components of α_i are parallel. Since $\tau(T_1) = T_2$ all components of ∂S project to isotopic loops in ∂W thus S has invariant slope for the cover $\tilde{W}\to M$. This gives:

Case (i) If S is not the fiber of a fibration of $\tilde{W}$ then the result follows from Lemma 7.

Thus we are left with the case that S is the fiber of a fibration of $\tilde{W}$. Let N be the $\mathbb{Z}_2$-universal covering of W. This is a regular covering and each component of ∂N is a two-fold cover of ∂W. We claim that the composition of coverings $N\to W\to M$ is regular.

Recall that a subgroup $H<G$ is *characteristic* if it is preserved by $Aut(G)$. The $\mathbb{Z}_2$-universal covering $N\to W$ corresponds to the characteristic subgroup $\pi_1 N<\pi_1 W$. The cover $W\to M$ is cyclic and so $\pi_1 W$ is normal in $\pi_1 M$. A characteristic subgroup of a normal subgroup is normal. Hence $\pi_1 N$ is also normal in $\pi_1 M$. This proves the claim. It follows that $N\to M$ is a ∂-cover. A pre-image, $\tilde{S}$, of S in N is a fiber of a fibration.

Case (ii) Suppose the one-dimensional vector space of $H_2(N,\partial N;\mathbb{R})$ spanned by $[\tilde{S}]$ is invariant under the group of covering transformations of $N\to M$.

Then, by Theorem 3, M is semi-fibered which contradicts our hypothesis. This completes case (ii). Therefore there is some covering transformation, σ, such that $\sigma_*[\tilde{S}]\neq\pm[\tilde{S}]$.

Because $\tilde{S}$ and $\sigma\tilde{S}$ are fibers, they both meet every boundary component of N. Since S has invariant slope for the cover $N\to M$ it follows that $\tilde{S}$ and $\sigma\tilde{S}$ have the same invariant slope for this cover.

Case (iii) Suppose S is a fiber and $[\partial\tilde{S}]\neq\pm\sigma_*[\partial\tilde{S}] \in H_1(\partial N)$.

Given a boundary component of N, there are integers a and b such that the class $a[\tilde{S}]+b\cdot\sigma_*[\tilde{S}]\in H_2(N,\partial N)$ is non-zero and represented by an essential surface G that misses this boundary component. Thus G is not a fiber of a fibration. Clearly G has invariant slope. The result now follows from Lemma 7 applied to the surface G in the ∂-cover N. This completes case (iii). The remaining case is:

Case (iv) S is a fiber and there is $\epsilon\in\{\pm 1\}$ with $\sigma_*[\partial\tilde{S}] = \epsilon\cdot[\partial\tilde{S}] \in H_1(\partial N)$.

Consideration of the homology exact sequence for the pair $(N,\partial N)$ shows $x = \sigma_*[\tilde{S}]-\epsilon\cdot[\tilde{S}]\in H_2(N,\partial N)$ is the image of some $y\in H_2(N)$. Using exactness of the sequence again it follows that $y+i_*H_2(\partial N)$ is not zero in $H_2(N)/i_*H_2(\partial N)$. Hence every filling of N produces a closed manifold with $\beta_2>0$. Infinitely many slopes on

∂M *lift* to slopes on ∂N. The result follows. This completes the proof of case (iv) and thus of Theorem 1. $\square$

Proof of corollary 2. Let $\eta(K)$ be an open tubular neighborhood of k. By hypothesis the knot exterior $M = N \setminus \eta(K)$ is irreducible. Every semibundle contains two disjoint compact surfaces whose union is non-separating, thus the first Betti number with mod-2 coefficients of a semi-bundle is at least 2. Because N is a homology sphere $H_1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, therefore M is not a semi-bundle. Since N is a homology sphere it, and therefore M, are orientable.

If M is a bundle with fiber F then, since N is a homology sphere, F has exactly one boundary component. Since k does not bound a disk in N it follows that $M \neq D^2 \times S^1$. The result now follows from Theorem 1. If M contains a closed essential surface then infinitely many fillings are Haken, [**?**, Theorem 2.4.2]. The remaining possibilities are that M is hyperbolic and not a bundle, or else Seifert fibered. The hyperbolic non-bundle case follows from [**?**].

This leaves the case that M is Seifert fibered. The manifold M is not a twisted I-bundle over the Klein bottle because the latter has mod-2 Betti number 2. The result now follows from Lemma 5. $\square$

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University of California,
Santa Barbara, CA, USA

Tufts University
Medford, MA, USA and

Université du Québec à Montréal
Montréal, Québec, Canada