Knot complements, hidden symmetries and reflection orbifolds

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Abstract

In this article we examine the conjecture of Neumann and Reid that the only hyperbolic knots in the 3-sphere which admit hidden symmetries are the figure-eight knot and the two dodecahedral knots. Knots whose complements cover hyperbolic reflection orbifolds admit hidden symmetries, and we verify the Neumann-Reid conjecture for knots which cover small hyperbolic reflection orbifolds. We also show that a reflection orbifold covered by the complement of an AP knot is necessarily small. Thus when K is an AP knot, the complement of K covers a reflection orbifold exactly when K is either the figure-eight knot or one of the dodecahedral knots.

1 Introduction

Commensurability is the equivalence relation on families of spaces or orbifolds determined by the property of sharing a finite degree cover. It arises in a number of areas of interest and is connected with other important concepts. For instance, two complete, finite volume, non-compact hyperbolic orbifolds are commensurable if and only if their fundamental groups are quasi-isometric [22].

In this paper we consider commensurability relations between the complements of hyperbolic knots in the 3-sphere. For convenience we say that two such knots are commensurable if their complements have this property. There are two conditions

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which play a pivotal role here, though more for their rarity than their regularity. The first is *arithmeticity*, a commensurability invariant which, among knots, is only satisfied by the figure-eight [20]. In particular, the figure-eight is the unique knot in its commensurability class. A basic result due to Margulis [15] states that the commensurability class of a non-arithmetic finite-volume hyperbolic orbifold has a minimal element. In other words, there is a hyperbolic orbifold covered by every member of the class. This holds for the complement of any hyperbolic knot K other than the figure eight. For such knots we use $\mathcal{O}_{full}(K)$ to denote the minimal orbifold in the commensurability class of $S^3 \setminus K$ and $\mathcal{O}_{min}(K)$ to denote the minimal orientable orbifold in the commensurability class of $S^3 \setminus K$. These orbifolds often coincide, but if they do not, $\mathcal{O}_{min}(K)$ is the orientation double cover of $\mathcal{O}_{full}(K)$.

The second key condition in the study of knot commensurability is the existence or not of *hidden symmetries* of a hyperbolic knot K. In other words, the existence or not of an isometry between finite degree covers of $S^3 \setminus K$ which is not the lift of an isometry of $S^3 \setminus K$. Equivalently, a hyperbolic knot complement admits hidden symmetries if and only if it covers some orbifold irregularly, or (for non-arithmetic knots) if and only if the minimal orientable orbifold in its commensurability class has a *rigid cusp*. (This means that the cusp cross section is a Euclidean turnover.) See [17, Proposition 9.1] for a proof of these equivalences. The latter characterisation shows that if K_1 and K_2 are commensurable, then K_1 admits hidden symmetries if and only if K_2 does.

The commensurability classification of hyperbolic knots which admit no hidden symmetries was extensively studied in [5], where it was shown that such classes contain at most three knots. Further, there are strong constraints on the topology of a hyperbolic knot without hidden symmetries if its commensurability class contains more than one knot. For instance, it is fibred.

To date, there are only three hyperbolic knots in S^3 which are known to admit hidden symmetries: the figure-eight and the two dodecahedral knots of Aitchison and Rubinstein described in [2]. Each is alternating, and the minimal orbifold in the commensurability class of the dodecahedral knot complements is a reflection orbifold (see §2). Using known restrictions on the trace field of a knot with hidden symmetries, Goodman, Heard, and Hodgson [10] have verified that these are the only examples of hyperbolic knots with 15 or fewer crossings which admit hidden symmetries. This lends numerical support to the following conjecture of W. Neumann and A. Reid.

Conjecture 1.1. (Neumann-Reid) *The figure-eight knot and the two dodecahedral knots are the only hyperbolic knots in S^3 admitting hidden symmetries.*

In this article we investigate the Neumann-Reid conjecture in the context of non-arithmetic knots K for which $\mathcal{O}_{full}(K)$ contains a reflection, a condition which implies that the knot admits hidden symmetries (Lemma 2.1). Our main results concern knot complements which cover hyperbolic reflection orbifolds, especially the complements of *AP knots*. A knot K is an AP knot if each closed essential surface in the exterior of K contains an accidental parabolic (see §3 for definitions). Small knots (i.e. knots whose exteriors contain no closed essential surfaces) are AP knots, but so are *toroidally alternating knots* [1], a large class which contains, for instance, all hyperbolic knots which are alternating, almost alternating, or Montesinos.

Theorem 1.2. *If the complement $S^3 \setminus K$ of a hyperbolic AP knot K covers a reflection orbifold $\mathcal{O}$, then $\mathcal{O}$ is a one-cusped tetrahedral orbifold and K is either the figure-eight knot or one of the dodecahedral knots.*

The cusp cross section of the orientable minimal orbifold in the commensurability class of a hyperbolic knot complement with hidden symmetries is expected to be $S^2(2,3,6)$.

Conjecture 1.3. (Rigid cusp conjecture) *The minimal orientable orbifold covered by a non-arithmetic knot complement with hidden symmetries has a rigid cusp of type $S^2(2,3,6)$.*

Here is a corollary of Theorem 1.2 that we prove in §3.

Corollary 1.4. *If the complement of an achiral, hyperbolic, AP knot K covers an orientable orbifold with cusp cross section $S^2(2,3,6)$, then K is the figure-eight or one of the dodecahedral knots.*

The proof of Theorem 1.2 follows from the fact (Proposition 3.2) that a reflection orbifold covered by a hyperbolic AP knot complement cannot contain a closed essential 2-suborbifold (i.e. an orbifold-incompressible 2-suborbifold which is not parallel to the cusp cross section - see §3 for definitions) together with the following result:

Theorem 1.5. *Suppose that K is a hyperbolic knot. If $S^3 \setminus K$ covers a reflection orbifold $\mathcal{O}$ which does not contain a closed, essential 2-suborbifold, then K is either the figure-eight knot and arithmetic or one of the dodecahedral knots and non-arithmetic.*

Since the figure-eight knot is the only arithmetic knot and the dodecahedral knots are not small ([4, Theorem 5] and, independently, [7, Theorem 8]), we obtain the following corollary.

Corollary 1.6. *No small, hyperbolic, non-arithmetic knot complement covers a reflection orbifold.* □

In [5] we proved that the complements of two hyperbolic knots without hidden symmetries are commensurable if and only if they have a common finite cyclic covering, that is, they are *cyclically commensurable*. Hence two commensurable hyperbolic knot complements are cyclically commensurable or admit hidden symmetries. Another consequence of Proposition 3.2 is that the minimal orientable orbifold $\mathcal{O}_{min}(K)$ in the commensurability class of the complement of a non-arithmetic hyperbolic AP knot K which admits hidden symmetries is small (Corollary 3.3). This allows us to extend results of N. Hoffman on small hyperbolic knots ([11, Theorems 1.1 and 1.2]) to the wider class of AP knots.

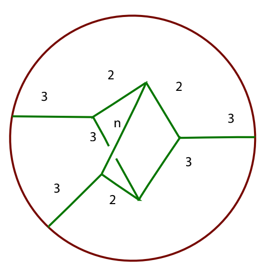
Theorem 1.7. *Let K be a non-arithmetic hyperbolic AP knot which admits hidden symmetries.*
(1) *$S^3 \setminus K$ is not cyclically commensurable with any other knot complement.*
(2) *If K admits a non-meridional, non-hyperbolic surgery, then $S^3 \setminus K$ admits no non-trivial symmetry and $\mathcal{O}_{min}(K)$ has an $S^2(2,3,6)$ cusp cross section. Further, $\mathcal{O}_{min}(K)$ does not admit a reflection.*

In our final result, we show that it is still possible to obtain strong restrictions on the combinatorics of the singular locus of $\mathcal{O}_{min}(K)$ if we replace the hypothesis that the minimal orbifold $\mathcal{O}_{full}(K)$ is a reflection orbifold by the weaker assumption that the orientable minimal orbifold $\mathcal{O}_{min}(K)$ admits a *reflection* (i.e. an orientation reversing symmetry with a 2-dimensional fixed point set).

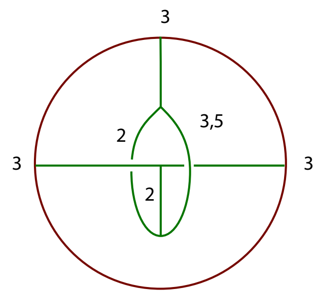
Theorem 1.8. *Let K be a non-arithmetic hyperbolic knot and suppose that $\mathcal{O}_{min}(K)$ does not contain an essential 2-suborbifold. Suppose further that $\mathcal{O}_{min}(K)$ admits a reflection. Then the $\mathcal{O}_{min}(K)$ is orbifold-homeomorphic to one of the following models:*

(a) *a one-cusped tetrahedral orbifold;*

(b) $Y333$ *with* $n = 2, 3, 4, 5$:



(c) XO :



The proof of Theorem 1.8 will show that the hypothesized reflection for the orbifolds listed in (b) or (c) is the one given by the obvious plane of symmetry in the accompaning figures. Thus the quotient has cusp cross-section $D^2(3;3)$ - a non-orientable, non-reflection Euclidean 2-orbifold.

Question 1.9. *Are any of the orbifolds listed above as type* (b) *or* (c) *in Theorem 1.8 covered by a knot complement?*

Conjecture 1.10. *No.*

Here is how the paper is organized. In section 2 we show that the presence of a reflection in the commensurability class of the complement of a hyperbolic knot K implies that K admits hidden symmetries (Lemma 2.1). Then we introduce the notion of a reflection orbifold and prove that for a non-arithmetic knot K, $\mathcal{O}_{full}(K)$ is a hyperbolic reflection orbifold as long its cusp cross section is a Euclidean reflection orbifold (Lemma 2.2). We also show that no hyperbolic knot complement covers a reflection orbifold regularly (Lemma 2.4(1)). In section 3 we consider hyperbolic AP knots whose complements cover reflection orbifolds (Theorem 1.2) and show that if such a knot admits hidden symmetries, then it cannot be cyclically commensurable with another knot (Theorem 1.7). In section 4 we determine those knots whose complements cover small hyperbolic reflection orbifolds (Theorem 1.5). Finally in section 5 we study the combinatorics of the minimal orbifolds in the commensurability classes of knot complements where the orientable commensurator quotient admits a reflection, but whose full commensurator quotient is not a reflection orbifold (Theorem 1.8).

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2 Knot complements and reflections

In this section we prove some general results about knots whose complements have reflections in their commensurability classes. For example, a hyperbolic amphichiral knot which admits hidden symmetries always has a reflection in its commensurability class. This is because the orientable commensurator quotient has a cusp cross section with three cone points, and therefore must admit a reflection. We assume that the reader is familiar with the terminology and notation from [5]. We refer to [6] for background information on the geometry and topology of low-dimensional orbifolds, and to [24] on how they relate to the knot commensurability problem.

If $K \subset S^3$ is a hyperbolic knot, there is a discrete subgroup $\Gamma_K \leq PSL_2(\mathbb{C})$, unique up to conjugation in $\mathrm{Isom}(\mathbb{H}^3)$, such that $\pi_1(S^3 \setminus K) \cong \Gamma_K$. We use M_K to denote the exterior of K.

Lemma 2.1. *Suppose a non-arithmetic hyperbolic knot complement $S^3 \setminus K = \mathbb{H}^3/\Gamma_K$ is commensurable with an orbifold $\mathcal{O}$ which admits a reflection symmetry. Then,*
(1) *$S^3 \setminus K$ admits hidden symmetries.*
(2) *$S^3 \setminus K$ contains a (possibly immersed) totally geodesic surface.*

Proof. First we prove (1). Let r be the reflection symmetry in $\mathcal{O}$. Then r is contained in the full commensurator $C(\Gamma_K) \leq \mathrm{Isom}(\mathbb{H}^3)$ of $\pi_1(S^3 \setminus K) \cong \Gamma_K \leq PSL_2(\mathbb{C})$. We claim that the full normalizer $N(\Gamma_K) \leq \mathrm{Isom}(\mathbb{H}^3)$ of Γ_K is a proper subgroup of $C(\Gamma_K)$. Otherwise, if the full commensurator is the full normalizer, then $r\Gamma_K r^{-1} = \Gamma_K$. But no hyperbolic knot complement is normalized by a reflection. This is because the normalizer quotient $N(\Gamma_K)/\Gamma_K$ is the group of isometries $Isom(S^3, K)$ and therefore K can be isotoped so as to be invariant under a reflection in a 2-sphere in S^3. Since the non-trivial knot cannot be contained in the reflection sphere, it must meet the sphere in two points, which implies it is a connect sum. Therefore, the knot complement admits a hidden reflection, and by [5, Theorem 7.2], the knot admits hidden symmetries. This proves (1).

For (2), suppose there is a reflection τ in $\pi_1^{orb}(\mathcal{O}_{full})$ though a plane P in $\mathbb{H}^3$. Then $\Gamma_K \cap \tau\Gamma_K\tau^{-1}$ is a finite index subgroup of $\Gamma_K$which is normalized by τ, as $\tau^2 = 1$. Thus $M_\tau = \mathbb{H}^3/(\Gamma_K \cap \tau\Gamma_K\tau^{-1})$ admits a reflection and the fixed point set of this reflection is a totally geodesic surface. As M_τ is a finite-sheeted cover of $S^3 \setminus K$, $S^3 \setminus K$ contains a totally geodesic surface. This surface may be immersed and may have cusps. □

Here are some definitions needed for our next result.

A *convex hyperbolic polyhedron* is the intersection of a finite number of half-spaces in $\mathbb{H}^n$.

A *hyperbolic reflection group* is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by a finite number of reflections.

Each hyperbolic reflection group $\Gamma \leq \text{Isom}(\mathbb{H}^n)$ has a fundamental domain which is a convex hyperbolic polyhedron $P \subseteq \mathbb{H}^n$. Further, Γ is generated by reflections in the faces of P. The quotient orbifold $\mathcal{O}_\Gamma = \mathbb{H}^n/\Gamma$ has underlying space P and singular set ∂P. The faces of P are reflector planes. See [8, Theorem 2.1] for a proof of these assertions.

A *hyperbolic reflection orbifold* is an orbifold $\mathcal{O}$ associated to the quotient of $\mathbb{H}^n$ by a hyperbolic reflection group.

Euclidean reflection orbifolds are defined similarly. We leave the details to the reader.

The orientation double cover $\widetilde{\mathcal{O}}$ of a one-cusped reflection n-orbifold $\mathcal{O}$ is obtained by doubling $\mathcal{O}$ along its reflector faces. Hence the interior of $\widetilde{\mathcal{O}}$ has underlying space $\mathbb{R}^n$ and singular set contained in a properly embedded hyperplane, which is the fixed point set of the reflection determined by the cover $\widetilde{\mathcal{O}} \to \mathcal{O}$. In the case of a 3-dimensional reflection orbifold, the singular set of $\widetilde{\mathcal{O}}$ is a trivalent "graph" with a finite number of vertices and a finite number of edges which are either lines or half-lines properly embedded in $\widetilde{\mathcal{O}}$, or compact intervals whose endpoints are vertices. (There are no circular edges since each loop contains a vertex.)

Lemma 2.2. *Let K be a non-arithmetic hyperbolic knot. Suppose that the cusp cross section of the full commensurator quotient $\mathcal{O}_{full}(K)$ of $S^3 \setminus K$ is a Euclidean reflection 2-orbifold. Then $\mathcal{O}_{full}(K)$ is a hyperbolic reflection orbifold.*

Proof. By Lemma 2.1(1), $S^3 \setminus K$ admits hidden symmetries and so by [5, Corollary 4.11], the minimal element $\mathcal{O}_{min}(K)$ in the orientable commensurability class of $S^3 \setminus K$ has underlying space a ball. Further, $\mathcal{O}_{min}(K)$ has a rigid cusp so its cusp cross section is a Euclidean 2-orbifold with 3 cone points. Thus it is of the form $S^2(2,3,6), S^2(2,4,4)$ or $S^2(3,3,3)$.

The full commensurator quotient $\mathcal{O}_{full}(K)$ of $\mathbb{H}^3$ is the quotient of $\mathcal{O}_{min}(K)$ by an orientation-reversing involution, so the cusp cross section of $\mathcal{O}_{full}(K)$ is the quotient of the cusp cross section of $\mathcal{O}_{min}(K)$ by an orientation-reversing involution. Since we have assumed that this quotient is a Euclidean reflection orbifold, it must have underlying space a triangle with boundary made up of three reflector lines and three corner-reflectors. Thus the fundamental group of the cusp cross section of $\mathcal{O}_{full}(K)$ is a triangle group. It follows that the peripheral subgroup is a group generated by reflections so in particular, any meridional class of the knot is a product of reflections in the full commensurator. Since any conjugate of a product of reflections is a product of reflections, the knot group, which is normally generated by the meridian, is generated by products of reflections.

Let $\Gamma_K, t_1\Gamma_K, t_2\Gamma_K, ...t_n\Gamma_K$ be the left cosets of Γ_K in the full commensurator. Since the one cusp of the knot complement covers the cusp of the commensurator quotient, the index of the covering restricted to the cusp is the same as the index of the cover. Thus we may take our t_i to be in the cusp group and therefore we may suppose that the t_i are products of reflections. It follows that every element in the full commensurator is a product of reflections in the full commensurator, so the group is generated by reflections. $\square$

Remark 2.3. Suppose that K is a non-arithmetic hyperbolic knot whose full commensurator contains an orientation reversing involution and whose orientable commensurator quotient has cusp cross section of the form $S^2(2,3,6)$. Since any involution of $S^2(2,3,6)$ fixes each of its cone points, the cusp cross section of $\mathcal{O}_{full}(K)$ is a Euclidean reflection 2-orbifold. Thus $\mathcal{O}_{full}(K)$ is a hyperbolic reflection orbifold by Lemma 2.2.

Lemma 2.4. *A hyperbolic reflection orbifold, and its orientation double cover, cannot be regularly covered by a knot complement.*

Proof. Suppose that a knot complement $S^3 \setminus K$ regularly covers a hyperbolic reflection orbifold $\mathcal{O}$ and consider the orientation double cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, which is also regularly covered by $S^3 \setminus K$. Since the interior of $\widetilde{\mathcal{O}}$ has underling space $\mathbb{R}^3$, its cusp cross section is $S^2(2,2,2,2)$. (The group of orientation-preserving isometries of a hyperbolic knot complement is cyclic or dihedral, by the positive solution of the Smith conjecture, so any orientable orbifold regularly covered by the knot complement has cusp cross section a torus or $S^2(2,2,2,2)$.) Thus $\widetilde{\mathcal{O}}$ is the quotient of $S^3 \setminus K$ by a dihedral group generated by a strong inversion of K and a cyclic symmetry σ of order $n \geq 1$ whose axes are disjoint from K.

We noted above that the singular set $\Sigma(\widetilde{\mathcal{O}})$ of $\widetilde{\mathcal{O}}$ is contained in the reflection 2-plane P of the cover $\widetilde{\mathcal{O}} \to \mathcal{O}$. Since the cusp cross section of $\widetilde{\mathcal{O}}$ is $S^2(2,2,2,2)$, the image in $\widetilde{\mathcal{O}}$ of the axis of the strong inversion is a pair of disjoint properly embedded real lines L_1 and L_2. This cannot be all of $\Sigma(\widetilde{\mathcal{O}}) \subset P$, as it does not form the edges of a polyhedron with one ideal vertex. Thus σ has order $n > 1$, and σ^k acts non-freely for some $k < n$. Since the quotient $(S^3 \setminus K)/\langle\sigma\rangle$ has singular set consisting of one or two circles, while, as we noted above, each loop in $\Sigma(\widetilde{\mathcal{O}})$ has a vertex, the axis of the strong inversion meets each of the axes of the cyclic symmetry in two points. It follows that $\Sigma(\widetilde{\mathcal{O}})$ is connected and is the union of L_1, L_2 and one arc for each of the circles in the singular set of $(S^3 \setminus K)/\langle\sigma\rangle$. The reader will verify that the hyperbolicity of $S^3 \setminus K$ forces $\Sigma(\widetilde{\mathcal{O}})$ to be the union of L_1, L_2 and two arcs, each running from L_1 to L_2. By construction, the sides of the quadrilateral labeled contained in L_1 and L_2 are disjoint and labeled 2. But then there is an essential $D^2(2,2)$ properly embedded in $\widetilde{\mathcal{O}}^{tr}$, the truncation of $\widetilde{\mathcal{O}}$ along an $S^2(2,2,2,2)$ cusp cross section, contrary to the hyperbolicity of $\widetilde{\mathcal{O}}$. This completes the proof. $\square$

3 AP knots

The main result of this section is given by Proposition 3.2, which allows us to reduce the proof of Theorem 1.2 to that of Theorem 1.5. We begin with some definitions.

Definition 3.1. *Let M be a compact 3-manifold with incompressible boundary.*
(1) *An* **accidental parabolic** *of an essential surface S in M is an essential loop on S which is homotopic in M to a peripheral curve of M.*
(2) *An* **AP knot** *is a knot K in S^3 such that any closed essential surface in the exterior of K contains an accidental parabolic.*

The class of AP knots includes *small knots* (i.e. knots whose exteriors contain no closed essential surfaces) and toroidally alternating knots, a class which contains all alternating knots, all almost alternating knots, and all Montesinos knots. See [1].

Let K be an AP knot. If S is a closed essential surface in the exterior M_K of K and N is the component of M_K cut open along S which contains ∂M_K, then the annulus theorem implies that there is an essential annulus properly embedded in N with boundary the union of an essential simple closed curve on S and an essential simple closed curve on ∂M_K. In the case that K is a toroidally alternating knot, we can take the essential simple closed curve on ∂M_K to be a meridional curve of K ([1]).

A 2-suborbifold $\mathcal{F}$ of a 3-orbifold $\mathcal{O}$ is *orbifold-incompressible* if there is no orbifold-essential curve on $\mathcal{F}$ that bounds an orbi-disc in $\mathcal{O}$, $\mathcal{F}$ is not a spherical orbifold which bounds an orbi-ball (i.e. an orbifold quotient of a 3-ball) , and $\mathcal{F}$ is finitely covered by a surface. It is *essential* if it is orbifold-incompressible and not boundary-parallel.

An obifold is *small* if it is irreducible and contains no closed, essential 2-suborbifold. Note that a small manifold cannot cover an orbifold which contains an essential closed 2-suborbifold.

Proposition 3.2. *If the complement of any prime AP knot K finitely covers an orientable orbifold $\mathcal{O}$ with rigid cusp, then $\mathcal{O}$ does not contain any closed essential orientable 2-suborbifold, and thus is small.*

Proof. We first note that the complement of a prime AP knot is atoroidal, so it is either a torus knot complement or a hyperbolic knot complement. If a torus knot complement cover an orbifold, the orbifold is Seifert-fibred. Since boundary components of an orientable Seifert fibered orbifold are tori or copies of $S^2(2,2,2,2)$ [6, §2.4.1], a torus knot complement never covers an orbifold with a rigid cusp. Thus we may assume that $S^3 \setminus K$ is hyperbolic. Let $\mathcal{O}^{tr}$ be the result of truncating $\mathcal{O}$ along a Euclidean turnover cross section of its cusp, and let M_K be the inverse image of $\mathcal{O}^{tr}$ in $S^3 \setminus K$. Then M_K is the exterior of K. Let $\pi : M_K \to \mathcal{O}^{tr}$ be the covering map. In order to obtain a contradiction, we assume that $\mathrm{int}(\mathcal{O}^{tr})$ contains a closed, essential, orientable, connected 2-suborbifold $\mathcal{F}$. Since $\mathcal{O}^{tr}$ is finitely covered by a hyperbolic knot exterior, $\mathcal{F}$ has negative Euler characteristic and separates $\mathcal{O}^{tr}$. The 2-suborbifold $\mathcal{F}$ splits $\mathcal{O}^{tr}$ into two compact 3-suborbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$, where $\partial\mathcal{O}_1 = \mathcal{F} \cup \partial\mathcal{O}^{tr}$. The preimage $F = \pi^{-1}(\mathcal{F})$ is a closed orientable (possibly disconnected) essential surface in ∂M_1 where $M_1 = \pi^{-1}(\mathcal{O}_1)$. Note that as $\pi^{-1}(\partial\mathcal{O}^{tr}) = \partial M_K$ is connected, M_1 is a compact, connected submanifold of M_K. In particular, it is the component of M_K split open along F with boundary $\partial M_1 = F \cup \partial M_K$.

A closed Euclidean 2-suborbifold is called *canonical* if it can be isotoped off any essential closed Euclidean 2-suborbifold. For instance, Euclidean turnovers are always canonical ([6, Remark, page 47]). By the JSJ theory of 3-orbifolds, any maximal collection of disjoint, canonical, essential, closed Euclidean 2-suborbifolds in a compact, irreducible 3-orbifold $\mathcal{O}$ is finite ([6, Theorem 3.11]) and unique up to isotopy ([6, Theorem 3.15]). It follows that each isotopy class of Euclidean turnovers in $\mathcal{O}$ is contained in this collection. If $\mathcal{O}$ contains no bad 2-suborbifolds, the solution of the geometrisation conjecture implies that either $\mathcal{O}$ is a closed *Sol* orbifold or the collection splits $\mathcal{O}$ into geometric pieces. See [6, §3.7].

Doubling $\mathcal{O}_1$ along its boundary produces a closed, connected, irreducible orbifold $D(\mathcal{O}_1)$ which is finitely covered by the double $D(M_1)$ of M_1. Thus $D(\mathcal{O}_1)$ is irreducible, contains no bad 2-suborbifolds, and contains an essential suborbifold of negative Euler characteristic (i.e. any component of $\mathcal{F}$). The latter fact implies that $D(\mathcal{O}_1)$ is not a *Sol* orbifold. It follows that a maximal collection of disjoint, canonical, essential, closed Euclidean 2-suborbifolds of $D(\mathcal{O}_1)$ splits it into geometric pieces. Since boundary components of a Seifert piece are tori or copies of $S^2(2,2,2,2)$ [6, §2.4.1], the pieces which are incident to $\partial\mathcal{O}_1$ must be hyperbolic.

The geometric splitting of $D(\mathcal{O}_1)$ lifts to a geometric splitting of $D(M_1)$ and, from the previous paragraph, the geometric pieces containing ∂M_K are hyperbolic. Thus ∂M_K is a JSJ torus in $D(M_1)$ and so any incompressible torus in $D(M_1)$ can be isotoped into its complement. On the other hand, since K is an AP knot, there is a properly embedded essential annulus A in M_1 running from $F = \partial M_1 \setminus \partial M_K$, to ∂M_K. The double $T = D(A)$ of this annulus is a torus in $D(M_1)$ which meets the torus ∂M_K along an essential simple closed curve, so that T is non-separating in $D(M_1)$. It follows that T is incompressible, as otherwise the irreducible manifold $D(M_1)$ would contain a non-separating 2-sphere. The double of the co-core of A is a simple closed curve on T which meets ∂M_K transversely in a single point, and hence is homologically dual to ∂M_K. But then T cannot be isotoped off of ∂M_K, which contradicts our observations above. Thus $\mathcal{O}$ does not contain any closed essential orientable 2-suborbifold, which completes the proof. □

Here is an immediate corollary of Proposition 3.2 and the fact that the minimal orientable orbifold in the commensurability class of a non-arithmetic hyperbolic knot complement with hidden symmetries has a rigid cusp ([17, Proposition 9.1]).

Corollary 3.3. *If the complement of a non-arithmetic hyperbolic AP knot K admits hidden symmetries, then the minimal orientable orbifold $\mathcal{O}_{min}(K)$ in its commensurability class is small.* □

Proof of Theorem 1.2 modulo Theorem 1.5. Let K be a hyperbolic AP knot and suppose that $S^3 \setminus K$ finitely covers a reflection orbifold $\mathcal{O}$. We claim that the orientation double cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, which is also covered by $S^3 \setminus K$, has a rigid cusp. Suppose otherwise and note that as the cusp cross section of $\mathcal{O}$ has underlying space a disk, the cusp cross section of $\widetilde{\mathcal{O}}$ is necessarily $S^2(2,2,2,2)$, and so the cover $S^3 \setminus K \to \widetilde{\mathcal{O}}$ is regular ([20, §6.2]), contrary to Lemma 2.4. Thus $\widetilde{O}$ has a rigid cusp, and by Proposition 3.2 it cannot contain a closed essential orientable 2-suborbifold. Then $\mathcal{O}$ cannot contain a closed essential 2-suborbifold. It follows that $S^3 \setminus K$ must cover a small reflection orbifold. Theorem 1.2 now follows from Theorem 1.5, whose proof is contained in §4. □

The proof of Corollary 1.4 follows from Theorem 1.2 and Remark 2.3.

Proof of Corollary 1.4. Since the complement of K covers an orientable orbifold with cusp cross section $S^2(2,3,6)$, its minimal orientable orbifold $\mathcal{O}_{min}(K)$ has cusp cross section $S^2(2,3,6)$. Moreover K admits an orientation reversing symmetry, hence its full commensurator is strictly bigger than its orientable commensurator and thus the full commensurator quotient $\mathcal{O}_{full}(K)$ is the quotient of the orientable minimal orbifold $\mathcal{O}_{min}(K)$ by an orientation reversing involution. Since any involution of the turnover $S^2(2,3,6)$ fixes each of its cone points, the cusp cross section of $\mathcal{O}_{full}(K)$ is a Euclidean reflection 2-orbifold. Thus $\mathcal{O}_{full}(K)$ is a hyperbolic reflection orbifold by Lemma 2.2 and the result follows from Theorem 1.2. □

Proof of Theorem 1.7. Hoffman proved part (1) of Theorem 1.7 for small hyperbolic knots ([11, Theorem 1.1]), though the smallness of $S^3 \setminus K$ is only used to deduce that the lattice $\Gamma_{\mathcal{O}_{min}(K)} \leq PSL_2(\mathbb{C})$, corresponding to the fundamental group of $\mathcal{O}_{min}(K)$, has integral traces. By Bass's theorem, the smallness of $\mathcal{O}_{min}(K)$ suffices to assure this last property, see [14, Theorem 5.2.2]. Therefore Corollary 3.3 allows us to extend Hoffman's result to the case of AP knots. This proves (1). Similarly the claims in the first sentence of Theorem 1.7(2) follows from the proof of [11, Theorem 1.2]. By Remark 2.3 and Theorem 1.2, if $\mathcal{O}_{min}(K)$ admits a reflection, it covers a reflection orbifold and so is one of the dodecahedral knots. But these both admit an orientation-preserving non-trivial symmetry. □

An *APM knot* is an AP knot K such that any closed essential surface in $S^3 \setminus K$ carries an essential curve which is homotopic to a meridian of K. Examples of

APM knots include all toroidally alternating knots. The next proposition puts constraints on the minimal orientable orbifold in the commensurability classes of certain APM knot complements.

Proposition 3.4. *If two distinct APM knot complements cover an orbifold $\mathcal{O}$ with a flexible cusp, then $\mathcal{O}$ is small.*

Proof. Let K_1, K_2 be the two APM knots. By the proof of Lemma 4.3 of [5] (see also [5, Remark 4.4]), their complements cover an orbifold with a torus cusp which, without loss of generality, we take to be $\mathcal{O}$. By [5, Corollary 4.11], $|\mathcal{O}|$ is the complement of a knot in a lens space, and by [9, Theorem 1.1], the knot complements $S^3 \setminus K_1, S^3 \setminus K_2$ cover $\mathcal{O}$ cyclically. Further, the images in $\mathcal{O}^{tr}$ of the two meridians μ_{K_1}, μ_{K_2} of the knots represent primitive classes $\bar{\mu}_{K_1}, \bar{\mu}_{K_2}$ of intersection number 1 on the boundary torus of $\mathcal{O}^{tr}$ ([5, Lemma 4.8]).

Consider a closed essential 2-suborbifold $\mathcal{S}$ contained in the interior of $\mathcal{O}$. Then $|\mathcal{S}|$ is a closed submanifold of the orientable 3-manifold $|\mathcal{O}|$, and as the latter is contained in a lens space, $|\mathcal{S}|$ is separating. Let $\mathcal{N}$ be the component of $\mathcal{O}^{tr}$ cut open along $\mathcal{S}$ which contains $\partial \mathcal{O}^{tr}$. Next let S_j be the inverse image of $\mathcal{S}$ in M_{K_j}, a closed, essential, separating surface. Let N_j be the component of M_{K_j} cut open along S_j which contains ∂M_{K_j}. Then $(N_j, \partial N_j) \to (N, \partial N)$ is a finite cyclic cover for both j.

Fix a base-point in $\partial \mathcal{O}^{tr}$ and lifts of it to ∂M_{K_1} and ∂M_{K_2}. By hypothesis there is an essential annulus in N_j which intersects ∂M_{K_j} in a meridional curve and S_j in an essential simple closed curve. In particular $\bar{\mu}_{K_2} \in \pi_1^{orb}(\mathcal{N})$ is represented by a loop in $\mathcal{N}$ of the form $\alpha\gamma\alpha^{-1}$ where α is a path in $\mathcal{N}$ connecting $\partial \mathcal{O}^{tr}$ to $\mathcal{S}$ and γ is a loop in $\mathcal{S}$. If n is the index of $\pi_1(S^3 \setminus K_1)$ in $\pi_1^{orb}(\mathcal{O})$, then $\bar{\mu}_{K_2}^n$ lifts to a class μ_2 in $\pi_1(N_1)$ represented by a loop in ∂M_{K_1} which is rationally independent of the class of μ_{K_1} in $H_1(\partial M_{K_2})$. Further, $\alpha\gamma^n\alpha^{-1}$ lifts to a loop of the form $\widetilde{\alpha}\beta\widetilde{\alpha}^{-1}$ where $\widetilde{\alpha}$ is a path in N_1 connecting ∂M_{K_1} to S_1 and β is a loop in S_1. Hence there is a singular annulus in N_1 which represents a homotopy between the loop μ_2 in ∂M_{K_1} and the loop β in S_1. On the other hand, by hypothesis there is an embedded annulus in N_1 representing a homotopy between μ_{K_1} and another loop in S_1. Hence there is a component of the characteristic Seifert pair of $(N_1, \partial N_1)$ homeomorphic to $S^1 \times S^1 \times I$ such that $S^1 \times S^1 \times \{0\}$ corresponds to ∂M_{K_1} and $S^1 \times S^1 \times \{1\}$ corresponds to a subsurface of S_1. But this impossible as it implies that S_1 has a component homeomorphic to a torus, contrary to the hyperbolicity of M_{K_1}. Thus $\mathcal{O}$ does not contain a closed essential 2-suborbifold. □

Corollary 3.5. *If a commensurability class contains two distinct hyperbolic APM knot complements, then the minimal orientable orbifold in this commensurability class is small.*

Proof. If the minimal orientable orbifold has a rigid cusp, the result follows from Proposition 3.2. Otherwise it has a flexible cusp, and we apply Proposition 3.4. □

4 Knot complements and reflection orbifolds

The goal of this section is to prove Theorem 1.5. We begin with a characterisation of the combinatorial type of small one-cusped reflection orbifolds.

We say that a orbifold $\mathcal{O}$ is a *one-cusped tetrahedral orbifold* if the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{O})$ is generated by reflections in the faces of a tetrahedron with one ideal point.

We say that $\mathcal{O}$ is a *one-cusped orbifold of quadrilateral type* if $\pi_1^{\mathrm{orb}}(\mathcal{O})$ is generated by reflections in the faces of a cone over a quadrilateral where the cone point is ideal.

Lemma 4.1. (The barrier lemma) *Let $\mathcal{O}$ be a 3-orbifold whose interior has underlying space an open* 3*-ball and which admits a complete finite volume hyperbolic structure. Let $\mathcal{O}^{tr}$ be the orbifold with boundary obtained by truncating the cusps of $\mathcal{O}$. Suppose that the singular set $\Sigma(\mathcal{O}^{tr})$ contains two disjoint* 1*-cycles rel boundary which are separated by a* 2*-sphere in $|\mathcal{O}|$. Then $\mathcal{O}$ contains a orbifold-incompressible 2-suborbifold.*

Proof. Call the two 1-cycles a and b. By hypothesis there is a 2-suborbifold $\mathcal{S}$ of $\mathcal{O}$ such that

1. $|\mathcal{S}| \cong S^2$;
2. $|\mathcal{S}|$ separates a and b;
3. $\mathcal{S}$ meets $\Sigma(\mathcal{O}^{tr})$ transversely and in the interior of its edges;
4. $\mathcal{S}$ has the minimal number of cone points among all 2-suborbifolds satisfying the first three conditions.

We will prove that $\mathcal{S}$ is is the desired 2-suborbifold. Recall that a 2-suborbifold $\mathcal{S} \subset (O)$ is *orbifold-incompressible* if there is no orbifold-essential curve on $\mathcal{S}$ that bounds an orbi-disc in $\mathcal{O}$, $\mathcal{S}$ is not a spherical orbifold which bounds an orbi-ball, and $\mathcal{S}$ is finitely covered by a surface.

Since $\mathcal{O}$ is finitely covered by a hyperbolic 3-manifold, $\mathcal{S}$ is finitely covered by a surface. The orbifold $\mathcal{S}$ cuts $\mathcal{O}^{tr}$ into two pieces, one of which contains $\partial\mathcal{O}^{tr}$. This piece cannot be an orbi-ball. The piece not containing $\partial\mathcal{O}^{tr}$ cannot be an orbi-ball either, since its singular set contains a cycle which does not meet $\mathcal{S}$ and the singular set of an orbi-ball is a single arc or a tripod. It remains to show that $\mathcal{S}$ does not admit a compressing orbi-disk.

Suppose that $\mathcal{S}$ admits a compressing orbi-disk $\mathcal{D}$. Since $|\mathcal{O}|$ is a ball, $|\mathcal{D}|$ intersects a and b zero times algebraically. Hence it is disjoint from a and b as it has at most one cone point. Since $\mathcal{S}$ has underlying space a sphere, $\partial\mathcal{D}$ bounds two discs on $\mathcal{S}$, each of which has at least two cone points as $\partial\mathcal{D}$ is essential on $\mathcal{S}$. One of the two 2-suborbifolds constructed from $\mathcal{D}$ and these two discs on $\mathcal{S}$ satisfies conditions (1), (2) and (3) and has fewer cone points, contrary to (4). Thus $\mathcal{S}$ is orbifold-incompressible. □

Lemma 4.2. *Let $\mathcal{O}$ be a one-cusped hyperbolic reflection orbifold which does not contain a closed orbifold-incompressible 2-suborbifold. Then $\mathcal{O}$ is either a one-cusped tetrahedral orbifold or a one-cusped orbifold of quadrilateral type.*

Proof. An alternative proof of this lemma can be deduced from Thurston's notes [23], particularly from the work contained in Chapter 13.

By hypothesis, $|\Sigma(\mathcal{O})| \cong \mathbb{R}^2$ and as the orientation double cover $\widetilde{\mathcal{O}} \to \mathcal{O}$ is obtained by doubling $\mathcal{O}$ along $\Sigma(\mathcal{O})$,

- the covering group of the cover $\widetilde{\mathcal{O}} \to \mathcal{O}$ is generated by a reflection r;
- $\Sigma(\widetilde{\mathcal{O}})$ is contained in the reflection plane of r;
- $|\widetilde{\mathcal{O}}| \cong \mathbb{R}^3$.

Further observe that a cross section of the cusp of $\mathcal{O}$ is a Euclidean reflection 2-orbifold and as such is generated by reflections in the faces of either a triangle or a quadrilateral. Thus the end of $\widetilde{\mathcal{O}}$ has cross section a Euclidean 2-orbifold with underlying space a 2-sphere and either three or four cone points. Let $\widetilde{\mathcal{O}}^{tr}$ denote the truncation of $\widetilde{\mathcal{O}}$. The boundary of $\widetilde{\mathcal{O}}^{tr}$ is the double of the cusp cross section of $\widetilde{\mathcal{O}}$ and so is either $S^2(2,3,6), S^2(2,4,4), S^2(3,3,3)$ or $S^2(2,2,2,2)$. The intersection of the reflection plane of r with $\widetilde{\mathcal{O}}^{tr}$ is a disc P.

Set $\Sigma = P \cap \Sigma(\widetilde{\mathcal{O}}) = \Sigma(\widetilde{\mathcal{O}}^{tr})$, since $\mathcal{O}$ is a reflection orbifold. By construction $\Sigma \cap \partial P$ consists of the three or four cone points which we call the *boundary vertices of* Σ. Further, Σ admits the structure of a graph whose vertices have valency 3 when contained in $\text{int}(P)$ (the *interior vertices*) and valency 1 otherwise. If Σ has a component contained in $\text{int}(P)$, it is separated from the other components of Σ by a circle embedded in $\text{int}(P)$. But this is impossible as otherwise there would be a reducing 2-sphere in $\widetilde{\mathcal{O}}$. Hence if Σ is not connected, there is a properly embedded disc D in $\widetilde{\mathcal{O}}^{tr}$ whose intersection with P is a properly embedded arc which separates P into two pieces, each containing components of Σ. Now ∂D splits the 2-sphere $|\partial\widetilde{\mathcal{O}}^{tr}|$ into two discs, each containing at least one point. In fact each contains two cone points as otherwise $\widetilde{\mathcal{O}}$ would contain a bad 2-suborbifold. But then ∂D is essential in $\partial\widetilde{\mathcal{O}}^{tr}$, which is impossible. Thus Σ is connected.

Suppose that Σ contains an arc a connecting two points on ∂P and an absolute cycle b which is disjoint from a. It is easy to see that there is a 2-sphere in $|\widetilde{\mathcal{O}}|$ which separates a and b and therefore $\widetilde{\mathcal{O}}$ contains an orbifold-incompressible 2-suborbifold by Lemma 4.1, contrary to our hypotheses. Thus each (absolute) cycle in Σ contains at least one vertex of any arc in Σ connecting two points of ∂P.

Next we observe that as all interior vertices of Σ have valency 3, there is a unique outermost embedded arc in Σ connecting any two of its boundary vertices which are adjacent on $\partial P \cong S^1$. Note that distinct outermost arcs which share an endpoint share the edge connecting that endpoint to an interior vertex. Number these outermost arcs $a_1, \ldots, a_n$ where $n = 3$ or $n = 4$ and a_i shares exactly one endpoint with a_{i+1}, where the indices are taken $\mod n$.

We claim each a_i has three edges. Indeed, it is easy to construct a boundary compressing orbi-disc in $\widetilde{\mathcal{O}}^{tr}$ if some a_i has two edges. Suppose then that there are more than three on some a_i. Then there are at least three interior vertices of a_i, say v_1, v_2 and v_3, where v_1 and v_3 are adjacent to $\Sigma \cap \partial P$ and v_2 is not. Consider the edge e_1 of Σ incident to v_2 but not contained in a_i. Since the interior vertices of Σ have valency 3, there is an oriented edge-path $e_1, e_2, \ldots, e_m$ in Σ such that

- e_k and e_{k+1} are distinct edges for each k ;
- it connects a_i to some a_j (it is possible that $i = j$);
- $e_2, \ldots, e_{m-1}$ are disjoint from $a_1 \cup \ldots \cup a_n$.

It is easy to see that there is an $l \neq i, j$ such that a_l is disjoint from an absolute cycle contained in $a_i \cup e_1 \cup \ldots \cup e_m \cup a_j$, contrary to what we deduced above. Thus each a_i has exactly three edges and the theorem follows quickly from this observation. □

We can now prove Theorem 1.5.

Proof of Theorem 1.5. First we show that $\mathcal{O}$ is a one-cusped tetrahedral orbifold. By Lemma 4.2 we need only show that $\mathcal{O}$ is not a one-cusped orbifold of quadrilateral type. Suppose otherwise. Then the cusp cross section of the orientation double cover $\widetilde{\mathcal{O}} \to \mathcal{O}$ is $S^2(2,2,2,2)$. The cover $S^3 \setminus K \to \mathcal{O}$ factors through $\widetilde{\mathcal{O}}$ so by [20, §6.2], the induced cover $S^3 \setminus K \to \widetilde{\mathcal{O}}$ is regular and dihedral. This contradicts Lemma 2.4. Thus $\mathcal{O}$ is a one-cusped tetrahedral orbifold.

The cusp cross section of $\widetilde{\mathcal{O}}$ is either $S^2(2,3,6)$, $S^2(3,3,3)$ or $S^2(2,4,4)$. Our strategy is to determine the indices of the singular locus of $\widetilde{\mathcal{O}}$. Here is an immediate constraint they satisfy.

(1) The link of each interior vertex is a spherical 2-orbifold.

Hoffman [11, Proposition 4.1] gives further restrictions on $H_1(\widetilde{\mathcal{O}})$ which hold for any orbifold covered by a knot complement. Namely:

(2) $H_1(\widetilde{\mathcal{O}})$ is a quotient of $\mathbb{Z}/2\mathbb{Z}$ if $\widetilde{\mathcal{O}}$ has cusp cross section $S^2(2,3,6)$.

We consider each possible cusp cross section in turn. Call an edge of $\Sigma(\widetilde{\mathcal{O}})$ *peripheral* if it is one of the non-compact edges which is properly embedded in the cusp of $\widetilde{\mathcal{O}}$.

First suppose that the cusp cross section is $S^2(2,3,6)$. Since $\mathcal{O}$ is tetrahedral, the indices on the edges of the interior triangle of the singular set $\Sigma(\widetilde{\mathcal{O}})$ determine the orbifold. By restriction (1), the two edges of the triangle meeting the peripheral edge of $\Sigma(\widetilde{\mathcal{O}})$ labeled 6 are labeled 2. Similarly the third edge is labeled $2, 3, 4$, or 5. The third edge cannot be labeled 2 or 4 as otherwise $H_1(\widetilde{\mathcal{O}})$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, violating restriction (2) above. If it is labeled 3, $\mathcal{O}$ is an arithmetic orbifold and is covered by the figure-eight knot complement, and hence K is the figure-eight knot (combine [18, §2] and [20]). If the third edge is labeled 5, $\widetilde{\mathcal{O}}$ is the minimal orbifold in the commensurability class of the dodecahedral knot complements [17, §9]. By the main result of [12], K is one of the dodecahedral knots. In both cases $\widetilde{\mathcal{O}}$ is a minimal element of this orientable commensurability class.

If the cusp-cross section is $S^2(3,3,3)$, restriction (1) implies that two of the edges of the interior triangle are labeled 2. But then $\widetilde{\mathcal{O}}$ admits a reflection with quotient the orientation double cover of a one-cusped tetrahedral orbifold and this double cover has cusp cross section $S^2(2,3,6)$. Thus K is the figure-eight knot, or one of the dodecahedral knots, see above.

Finally suppose that the cusp cross section is $S^2(2,4,4)$. We will show that this assumption leads to a contradiction.

Without loss of generality we can suppose that $\mathcal{O}$ does not cover another reflection orbifold non-trivially. By restriction (1), at least one of the two edges of the triangle meeting a peripheral edge labeled 4 is labeled 2. The two edges of the triangle incident to the peripheral edge labeled 2 cannot both have the same label as otherwise $\widetilde{\mathcal{O}}$ would admit a reflection with quotient the orientation double cover of a one-cusped tetrahedral orbifold. In particular one of these edges has label 3 or more. By restriction (1) it has label 3, and then it is easy to see that the other two edges of the triangle are labeled 2. Thus $\mathcal{O}$ is arithmetic and is the minimal element in the commensurability class of the Whitehead link (combine [18, §2] and [25, Example 1]), and therefore is different from the class of the figure-eight knot complement. But then by [20], there are no knots in this commensurability class of $\mathcal{O}$. This proves (2). □

5 Commensurators containing a reflection

This section is devoted to the proof of Theorem 1.8. Because of Proposition 3.2 this theorem applies to a hyperbolic AP knot K whose complement's orientable commensurator quotient admits a reflection, but whose full commensurator quotient is not a reflection orbifold. It gives strong restrictions on the topology and combinatorics of $\mathcal{O}_{min}(K)$.

Proof of Theorem 1.8. By Lemma 2.1(1), $S^3 \setminus K$ admits hidden symmetries and so by [5, Corollary 4.11], $\mathcal{O}_{min}(K)$ has underlying space a ball. Further, $\mathcal{O}_{min}(K)$

has a rigid cusp so its cusp cross section is of the form $S^2(2,3,6), S^2(2,4,4)$ or $S^2(3,3,3)$.

The full commensurator quotient $\mathcal{O}_{full}(K)$ of $S^3 \setminus K$ is the quotient of $\mathcal{O}_{min}(K)$ by the hypothesized reflection $r : \mathcal{O}_{min}(K) \to \mathcal{O}_{min}(K)$. If each of the three peripheral (i.e. non-compact) edges of the ramification locus of is invariant under r, then the cusp cross section of $\mathcal{O}_{min}(K)$ is a reflection orbifold and therefore Lemma 2.2 implies that $\mathcal{O}_{full}(K)$ is a reflection orbifold. By Theorem 1.5, K is one of the dodecahedral knots, so $\mathcal{O}_{min}(K)$ is a one-cusped tetrahedral orbifold (cf. [17, §9]). Thus $\mathcal{O}_{min}(K)$ satisfies (a). Assume below that this does not happen. Then r leaves exactly one of the peripheral edges of $\Sigma(\mathcal{O}_{min})$ invariant. It follows that cusp cross section of $\mathcal{O}_{min}$ is either $S^2(2,4,4)$ or $S^2(3,3,3)$ (cf. Remark 2.3). In the first case r preserves the peripheral edge labeled 2.

Denote the truncation of $\mathcal{O}_{min}(K)$ by $\mathcal{O}_{min}(K)^{tr}$ and let P be the intersection of $\mathcal{O}_{min}(K)^{tr}$ with the reflection plane of r. Then $|\mathcal{O}_{min}(K)^{tr}|$ is homeomorphic to a 3-ball and P a properly embedded disc in $|\mathcal{O}_{min}(K)^{tr}|$. By assumption, the circle ∂P contains exactly one of the cone points of the cusp cross section of $\mathcal{O}_{min}(K)$ contained in $\partial \mathcal{O}_{min}(K)^{tr}$. Further, the two remaining cone points both have order 3 or both have order 4.

An open regular r-invariant neighborhood of $P \cup \partial \mathcal{O}_{min}(K)^{tr}$ in $\mathcal{O}_{min}(K)^{tr}$ has complement consisting of two connected orbifolds $\mathcal{B}_L$ and $\mathcal{B}_R$. By construction $r(\mathcal{B}_L) = \mathcal{B}_R$. Both $\mathcal{B}_L$ and $\mathcal{B}_R$ have underlying space a 3-ball.

Set $\Sigma = \Sigma(\mathcal{O}_{min}(K)) \cap \mathcal{O}_{min}(K)^{tr}$. Let $\mathcal{S}_L = \partial \mathcal{B}_L$ and observe that $|\Sigma \cap \mathcal{S}_L| \geq 2$. For if $|\Sigma \cap \mathcal{S}_L| = 0$, the singular set of $\mathcal{O}_{min}(K)$ is contained in P, contrary to our assumptions, and if $|\Sigma \cap \mathcal{S}_L| = 1$, $\mathcal{S}_L$ would be a bad 2-suborbifold of $\mathcal{O}_{min}(K)$.

We claim that $|\Sigma \cap \mathcal{S}_L| \leq 3$. Suppose otherwise. Then $\mathcal{S}_L$ has four or more cone points, at least one of which has order 3 or 4, so it is a hyperbolic 2-orbifold. By hypothesis, $\mathcal{S}_L$ is orbifold-compressible in $\mathcal{O}_{min}(K)$. Thus there is a compressing orbi-disc $\mathcal{D}$ which meets the singular locus of $\mathcal{O}_{min}(K)^{tr}$ in at most one cone point. Assume that $\mathcal{D}$ is chosen to minimize the number of components of $\mathcal{D} \cap P$ and consider a 2-suborbifold $\mathcal{I}$ of $\mathcal{D}$ whose boundary is a component $\mathcal{D} \cap P$ which is innermost on $\mathcal{D}$. Then $\mathcal{I} \cup r(\mathcal{I})$ is a 2-suborbifold of $\mathcal{O}_{min}(K)$ with underlying space S^2. Since $\mathcal{D}$, and therefore $\mathcal{I}$, has at most one cone point, $\mathcal{I} \cup r(\mathcal{I})$ has at most two cone points. It cannot have one as otherwise $\mathcal{O}_{min}(K)$ would contain a bad 2-suborbifold. Thus it has zero or two cone points, and if two, they are cone points of the same order. It follows that $\mathcal{I} \cup r(\mathcal{I})$ is a spherical 2-suborbifold of $\mathcal{O}_{min}(K)$ and hence must bound an orbi-ball. But then we can reduce the number of components of $\mathcal{D} \cap P$, contradicting our assumptions. Thus $\mathcal{D}$ is disjoint from P. Since $\mathcal{S}_L$ is one boundary component of a regular neighborhood of $\partial \mathcal{O}_{min}(K)^{tr} \cup P$, removing $\mathcal{B}_L$ from the component of $\mathcal{O}_{min}(K)^{tr} \setminus P$ which contains it results in a product orbifold $\mathcal{P}_L = \mathcal{S}_L \times (0,1)$. Hence any compressing orbi-disc $\mathcal{D}$ for $\mathcal{S}_L$ is contained in $\mathcal{B}_L$.

Consider a compressing orbi-disc $\mathcal{D}$ for $\mathcal{S}_L$ in $\mathcal{B}_L$. Since $\mathcal{P}_L$ is a product, it contains an annulus A cobounded by $\partial\mathcal{D}$ and a simple closed curve on P. Then $\mathcal{D} \cup A \cup r(A) \cup r(\mathcal{D})$ is a 2-suborbifold with either zero or two cone points. In either case it must be a spherical 2-suborbifold bounding an orbi-ball. But then $\partial\mathcal{D}$ would be inessential in $\mathcal{S}_L$, contrary to our assumptions. We conclude that $\mathcal{S}_L$, and therefore $\mathcal{S}_R = r(\mathcal{S}_L)$, has at most three cone points. Since $\mathcal{O}_{min}(K)$ does not contain an orbifold-incompressible 2-suborbifold, both $\mathcal{B}_L$ and $\mathcal{B}_R$ must be orbi-balls. We divide the remainder of the proof into two cases.

Case 1. $|\Sigma \cap S_L| = 3$.

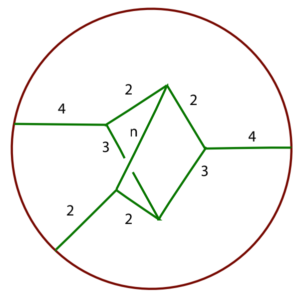
In this case, the singular loci of the orbi-balls $\mathcal{B}_L$ and of $\mathcal{B}_R$ are tripods. By construction, the endpoints of two edges of each of these tripods lie in P while the endpoints of the third edges are cone points of equal order (3 or 4) on $\partial\mathcal{O}^{tr}_{min}$. The union of the two tripods contains a circular 1-cycle a_0 homeomorphic to a circle and two relative 1-cycles a_1 and a_2 homeomorphic to intervals and properly embedded in $\mathcal{O}_{min}(K)$.

Consider $\Sigma \cap P$. There are two points x_1, x_2 in $\Sigma \cap P$ which correspond to the intersection of the legs of the tripods with P. Note that each of these two points may or may not be a vertex of $\Sigma(\mathcal{O}_{min}(K))$. If we remove the elements of $\{x_1, x_2\}$ which are not vertices of $\Sigma(\mathcal{O}_{min}(K))$ from Σ, what remains of $\Sigma \cap P$ inherits the structure of a graph from $\Sigma(\mathcal{O}_{min}(K))$ whose vertices have valency 1 or 3. Those of valency 1 are x_0, the unique cone point of $\partial\mathcal{O}^{tr}_{min}$ contained in ∂P, and whichever of x_1, x_2 is a vertex of $\Sigma(\mathcal{O}_{min}(K))$.

Suppose that there is a circle b in the graph $\Sigma \cap P$. Let E be the interior of the disc in P that it bounds. If E contains both x_1 and x_2 or neither of them, then it is easy to construct a 2-sphere in the interior of $|\mathcal{O}_{min}(K)|$ which separates a_0 and b. If it contains exactly one of x_1 and x_2, then one of the relative 1-cycles a_1, a_2 can be separated from b by a 2-sphere. Each of these possibilities contradicts the barrier lemma (Lemma 4.1). Thus $\Sigma \cap P$ is a finite union of trees. Each tree containing an edge has at least two extreme vertices, each extreme vertex has valency 1, and the vertices of valency 1 of $\Sigma \cap P$ are contained in $\{x_0, x_1, x_2\}$. Since all other vertices have valency 3 and the tree T_0 containing x_0 has at least one edge, it is easy to argue that either

- T_0 is an interval with boundary $\{x_0, x_1\}$, say, and $\mathcal{O}_{min}(K)$ is a one-cusped tetrahedral orbifold, or
- T_0 is a tripod with extreme vertices $\{x_0, x_1, x_2\}$ and the underlying graph of $\Sigma(\mathcal{O}_{min}(K))$ is as depicted in (b) and (c) of the statement of the theorem.

In the first case, $\mathcal{O}_{min}(K)$ satisfies (a). Suppose that the second case arises and recall that we noted above that the cusp cross section of $\mathcal{O}_{min}(K)$ is $S^2(3,3,3)$ or $S^2(2,4,4)$. The requirements that the interior vertices of $\Sigma(\mathcal{O}_{min}(K))$ correspond to spherical quotients and that $\mathcal{O}_{min}(K)^{tr}$ has no orientation-preserving symmetry allows us to determine the local isometry groups of $\mathcal{O}_{min}(K)$ and we conclude that (b) occurs when the cusp cross section is $S^2(3,3,3)$. When the cusp cross section is $S^2(2,4,4)$ the same analysis shows that we have the following orbifold:



This orbifold is double covered by an orbifold which has a $S^2(2,2,2,2)$ cusp and which has a loop labelled 3, which is the full pre-image of the two arcs labelled 3. The associated order 3 element is not in the normal closure of the cusp subgroup, so it cannot be covered by a knot complement, [5, Proof of Corollary 4.11] or [11, Proposition 2.3].

Case 2. $|\Sigma \cap S_L| = 2$.

In this case, both $\mathcal{B}_L$ and $\mathcal{B}_R$ are quotients of a ball by a finite cyclic rotational action, and $\Sigma \cap P$ has one point x_1 where Σ meets P transversely. The vertices of $\Sigma \cap P$ of valency 1 include x_0 and are contained in $\{x_0, x_1\}$. All other vertices have valency 3. Let Σ_0 be the component of $\Sigma \cap P$ containing x_0. If Σ_0 is a tree, it must be an interval (cf. the previous paragraph) with boundary $\{x_0, x_1\}$. Any other component of $\Sigma \cap P$ would contain 1-cycles, which is easily seen to contradict the barrier lemma. Thus $\Sigma \cap P = \Sigma_0$, so that Σ is a tripod with Euclidean labeling, which is impossible.

Suppose then that Σ_0 contains circular 1-cycles. Any other component of $\Sigma \cap P$ has at most one extreme vertex and so must contain circular 1-cycles as well, which is easily seen to contradict the barrier lemma. Thus $\Sigma \cap P = \Sigma_0$.

Fix a circular 1-cycle b in Σ_0 and observe that if b contained x_1, then x_1 would be a vertex of $\Sigma(\mathcal{O}_{min}(K))$ of valency at least 4, which is impossible. It follows from the barrier lemma that the interior of the disc in P bounded by b must contain x_1. If there is another circular 1-cycle b' in Σ_0, the barrier lemma implies that b and b' have a non-empty intersection, and using the fact that their vertices have valency 3 in P, they share at least one edge. It is then easy to see that Σ_0 contains a third circular 1-cycle which does not enclose x_1, which contradicts the barrier lemma. Thus b is the only circular 1-cycle in Σ_0, and it is easy to argue that Σ_0 satisfies one of the following two scenarios:

- b consists of a single edge with end-point x_2 and Σ_0 is the union of b and an edge connecting x_2 to x_0. To avoid contradicting the barrier lemma, x_1 must be in the interior of the disk bounded by b, and will be isolated in $\Sigma \cap P$.
- b is the union of two edges and Σ_0 is the union of b and two other edges - one connecting x_0 to one vertex of b and the other connecting x_1 to the other vertex of b.

We can rule out the first possibility since it would imply that $\mathcal{O}_{min}(K)$ admits a rotational symmetry of angle π. Thus the second possibility holds and therefore the underlying graph of $\Sigma(\mathcal{O}_{min}(K))$ is as depicted in (c). The requirements that the interior vertices of $\Sigma(\mathcal{O}_{min}(K))$ correspond to spherical quotients and that $\mathcal{O}_{min}(K)^{tr}$ has no orientation-preserving symmetry determines the local isometry groups. We conclude that the cusp cross section is $S^2(3,3,3)$ and that the labels on the edges of $\Sigma(\mathcal{O}_{min}(K))$ are as given in (c). This completes the proof. $\square$

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